### Shore's computational reverse mathematics

Benedict Eastaugh

University of Bristol

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### A foundational dialectic

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- I How do we know what mathematics we're giving up?

### Reverse mathematics can help

If we formalise our foundation in *second order arithmetic*, results in reverse mathematics will let us know which theorems we're entitled to assert and which remain out of reach.

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This is done by proving *equivalences* between such theorems and subsystems of second order arithmetic, over a weak base theory.

### Syntax and semantics of second order arithmetic

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 $\rm L_2\text{-}structures$  are models of the first order language of arithmetic extended with a collection of sets for the second order variables to range over:

$$\mathcal{M} = \langle M, S, +, \cdot, <, 0, 1 \rangle$$

where  $S \subseteq \mathcal{P}(M)$ .

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$$(0 \in X \land \forall n (n \in X \rightarrow n+1 \in X)) \rightarrow \forall n (n \in X).$$

• Comprehension scheme:

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

for all  $\varphi$  with X not free.

## Subsystems of second order arithmetic

Subsystems of  $Z_2$  are primarily obtained by restricting the comprehension scheme to particular syntactically defined subclasses.

Subsystems of $Z_2$	Defining conditions
RCA <sub>0</sub>	Recursive $(\Delta_1^0)$ comprehension
WKL <sub>0</sub>	RCA0 plus weak König's lemma
ACA <sub>0</sub>	Arithmetical comprehension
ATR <sub>0</sub>	ACA <sub>0</sub> plus arithmetical transfinite recursion
$\Pi_1^1 - CA_0$	$\Pi_1^1$ comprehension

### Foundational programmes and the Big Five

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Foundational programmes	Subsystems of $Z_2$
Constructivism	RCA <sub>0</sub>
Finitistic reductionism	WKL <sub>0</sub>
Predicativism	ACA <sub>0</sub>
Predicative reductionism	ATR <sub>0</sub>
Impredicativity	$ATR_0$ $\Pi_1^1$ -CA <sub>0</sub>

# Varieties of induction

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Contrast this with the *induction scheme*, each instance of which is a theorem of  $Z_2$ :

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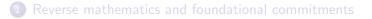
$$(\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n \varphi(n)$$

for all formulae  $\varphi$  in the language of second order arithmetic.

Weaker forms of induction can be obtained by restricting this scheme to particular classes such as the  $\Sigma_1^0$  formulae.

Induction axioms and subsystems of Z<sub>2</sub>

$\boldsymbol{\Sigma}_1^0$ induction	Induction axiom	Full induction scheme
RCA <sub>0</sub>		RCA
WKL <sub>0</sub>		WKL
	ACA <sub>0</sub>	ACA
	ATR <sub>0</sub>	ATR
	$ATR_0$ $\Pi_1^1$ -CA <sub>0</sub>	$\Pi_1^1$ -CA



3 Computable entailment and justification

Developed by Richard Shore in two recent papers (Shore 2010, 2011), computational reverse mathematics draws on recursion theory rather than proof theory.

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It has a two-fold motivation:

• Giving an account of reverse mathematics which most mathematicians will find natural, in computational and construction-oriented terms.

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It has a two-fold motivation:

- Giving an account of reverse mathematics which most mathematicians will find natural, in computational and construction-oriented terms.
- Extending reverse mathematical analysis from countable structures to uncountable ones.

Can computational reverse mathematics be used to carry out the foundational analysis outlined at the beginning?

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To answer this, we first need to look at the details of Shore's programme.

### $\omega\text{-models}$

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- Second order part C ⊆ P(ω) closed under particular recursion-theoretic operations.
- Closure under more operations  $\Leftrightarrow$  model of stronger theories.

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### Definition (Turing ideal)

Let C be a nonempty subset of  $\mathcal{P}(\omega)$  closed under Turing reducibility and recursive joins. Then we call C a *Turing ideal*.

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Let C be a nonempty subset of  $\mathcal{P}(\omega)$  closed under Turing reducibility and recursive joins. Then we call C a *Turing ideal*.

A set X is *Turing reducible* to a set Y,  $X \leq_T Y$ , iff there is a Turing machine with an oracle for Y which computes X.

The *recursive join* of two sets X and Y is given by

$$X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}.$$

Closure conditions and subsystems of  $Z_2$ 

Closure conditions	Subsystems of $Z_2$
Turing reducibility and recursive joins	RCA
Jockush–Soare low basis theorem	WKL
Turing jump	ACA
Hyperarithmetic reducibility	ATR
Hyperjump	$\Pi^1_1$ -CA

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These definitions extend to theories in the obvious way.

## Computable entailment is insensitive to induction

Because computable entailment only considers  $\omega$ -models, systems with restricted induction are computably equivalent to those with the full second order induction scheme.

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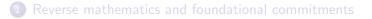
 $\begin{aligned} &\mathsf{RCA}_0 \equiv_{\mathcal{C}} \mathsf{RCA} \\ &\mathsf{WKL}_0 \equiv_{\mathcal{C}} \mathsf{WKL} \\ &\mathsf{ACA}_0 \equiv_{\mathcal{C}} \mathsf{ACA} \end{aligned}$ 

## Computable entailment is insensitive to induction

Because computable entailment only considers  $\omega$ -models, systems with restricted induction are computably equivalent to those with the full second order induction scheme.

 $RCA_0 \equiv_{\mathcal{C}} RCA$  $WKL_0 \equiv_{\mathcal{C}} WKL$  $ACA_0 \equiv_{\mathcal{C}} ACA$ 

Why is this problematic? Because the full second order induction scheme is proof-theoretically very strong.



2 Computational reverse mathematics



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Gödel's second incompleteness theorem shows that there is no such proof. Hilbert's programme therefore cannot be carried out in its entirety.

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Theorem (Sieg)

PRA proves that  $WKL_0$  is  $\Pi_2^0$ -conservative over PRA.

Theorem

 $\mathsf{WKL} \vdash \mathrm{Con}(\mathrm{PRA}).$ 

Theorem WKL ⊢ Con(PRA).

Con(PRA) is a  $\Pi_1^0$  statement not provable in PRA

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Theorem

 $WKL \vdash Con(PRA).$ 

 ${\rm Con(PRA)}$  is a  $\Pi^0_1$  statement not provable in  ${\rm PRA},$  so WKL is not  $\Pi^0_1\text{-}conservative over <math display="inline">{\rm PRA}$ 

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Theorem WKL  $\vdash$  Con(PRA).

Con(PRA) is a  $\Pi_1^0$  statement not provable in PRA, so WKL is not  $\Pi_1^0$ -conservative over PRA and therefore not finitistically reducible to it.

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But  $WKL_0 \models_{\mathcal{C}} WKL$ , so computable entailment does not preserve justification within the foundational programmes it seeks to analyse.

## Conclusion

Computational reverse mathematics doesn't respect the justificatory structure of foundational programmes.

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So whatever its merits, Shore's framework doesn't seem suitable for the kind of foundational analysis outlined at the beginning of the talk.

Thank you.

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### References

- R. A. Shore. Reverse Mathematics: The Playground of Logic. *The Bulletin of Symbolic Logic*, Volume 16, Number 3, 2010.
- R. A. Shore. Reverse mathematics, countable and uncountable: a computational approach. Manuscript, 2011.
- S. G. Simpson. Partial realizations of Hilbert's program. *The Journal of Symbolic Logic*, 53:349–363, 1988.